Recovering Coefficients in a System of Semilinear Helmholtz Equations from Internal Data

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May 19, 2023

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The inverse problem

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain with smooth boundary $\partial\Omega$.

- Model parameters: $\sigma(\mathbf{x})$ (conductivity), $\eta(\mathbf{x})$ (first-order susceptibility), $\gamma(\mathbf{x})$ (second-order nonlinear susceptibility), $\Gamma(\mathbf{x})$ (Grüneisen coefficient)
- Forward model (wave propagation with second harmonic generation):

$$
\begin{array}{rcll}
\Delta u + k^2(1+\eta)u + ik\sigma u & = & -k^2\gamma u^* v, & \text{in } \Omega \\
\Delta v + (2k)^2(1+\eta)v + i2k\sigma v & = & -(2k)^2\gamma u^2, & \text{in } \Omega \\
u = g, & & v = h, & \text{on } \partial\Omega
\end{array}
\tag{1}
$$

- Data (from thermo-acoustic tomography):

$$H(\mathbf{x}) = \Gamma(\mathbf{x})\sigma(\mathbf{x})(|u|^2 + |v|^2), \qquad \mathbf{x} \in \Omega$$

Inverse Problem

Determine the model parameters $(\Gamma, \sigma, \eta, \gamma)$ from the map

$$\Lambda_{\Gamma,\sigma,\eta,\gamma} : (g, h) \mapsto H.$$

Table of Contents

Table of Contents

Well-posedness of the forward model

Theorem

Let $\alpha \in (0,1)$. There exist $\varepsilon > 0$ and $\delta > 0$ such that for all $g, h \in \mathcal{C}^{2,\alpha}(\partial\Omega;\mathbb{C})$ with $\|g\|_{\mathcal{C}^{2,\alpha}(\partial\Omega)} < \varepsilon$ and $\|h\|_{\mathcal{C}^{2,\alpha}(\partial\Omega)} < \varepsilon$, the forward model (1) has a unique solution

$$(u,v) \in \{f \in \mathcal{C}^{2,\alpha}(\overline{\Omega};\mathbb{C}) : \|f\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \leq \delta\}^2.$$

Proof is a more-or-less straightforward application of the Banach fixed point theorem.

Table of Contents

Linearization

- **Observation**: when $g = h = 0$, we know that the solution to the forward model is $u = v = 0$, *regardless of the unknown coefficients* (η, σ, γ). This motivates linearizing around the boundary conditions $g = h = 0$.
- For a given small number $\varepsilon > 0$, let $(u_\varepsilon, v_\varepsilon)$ be the solution to the system

$$
\begin{aligned}
\Delta u_\varepsilon + k^2(1+\eta)u_\varepsilon + ik\sigma u_\varepsilon &= -k^2\gamma u_\varepsilon^* v_\varepsilon, \quad \text{in } \Omega \\
\Delta v_\varepsilon + (2k)^2(1+\eta)v_\varepsilon + i2k\sigma v_\varepsilon &= -(2k)^2\gamma u_\varepsilon^2, \quad \text{in } \Omega
\end{aligned}
$$

with boundary conditions

$$
u_\varepsilon = \varepsilon g, \qquad v_\varepsilon = \varepsilon h, \qquad \text{on } \partial\Omega \,. \tag{2}
$$

Denote the corresponding data by $H_\varepsilon := \Gamma\sigma(|u_\varepsilon|^2 + |v_\varepsilon|^2)$.
- When $\varepsilon = 0$, clearly $u_0 = v_0 = H_0 = 0$.
- Consider the derivatives $u^{(n)} := \partial_\varepsilon^n u_\varepsilon|_{\varepsilon=0}$, $v^{(n)} := \partial_\varepsilon^n v_\varepsilon|_{\varepsilon=0}$ and $H^{(n)} := \partial_\varepsilon^n H_\varepsilon|_{\varepsilon=0}$.

First-order linearization

- Taking the derivative of the forward model with respect to ε at $\varepsilon = 0$ yields

$$\begin{aligned}
\Delta u^{(1)} + k^2(1+\eta)u^{(1)} + ik\sigma u^{(1)} &= 0, \quad \text{in } \Omega \\
\Delta v^{(1)} + (2k)^2(1+\eta)v^{(1)} + i2k\sigma v^{(1)} &= 0, \quad \text{in } \Omega \\
u^{(1)} = g, \qquad v^{(1)} = h, & \qquad \text{on } \partial\Omega\,.
\end{aligned}$$

- Meanwhile, $H^{(1)} = 0$ and

$$H^{(2)} = 2\Gamma\sigma\left(u^{(1)*}u^{(1)} + v^{(1)*}v^{(1)}\right).$$

Theorem

All of the above derivatives exist.

Reconstruction of (Γ, σ, η)

- By taking $h = 0$, the problem reduces to reconstructing (Γ, σ, η) from the data
$$H = \Gamma\sigma|u|^2,$$
with the model for u given by
$$\begin{aligned} \Delta u + k^2(1+\eta)u + ik\sigma u &= 0, \quad \text{in } \Omega \\ u &= g, \quad \text{on } \partial\Omega\,. \end{aligned}$$

- When Γ and η are *known*, this problem was analyzed in Ammari et al. [1] (2012) and Bal et al. [4] (2011).

Reconstruction of (Γ, σ, η) (cont.)

- When Γ and η are *unknown*, use polarization identity. Let g_1 and g_2 be two incident sources. Measuring data corresponding to the illuminations g_1, g_2, $g_1 + g_2$ and $g_1 + ig_2$ gives us that the quantity

$$\Gamma\sigma u_1 u_2^* = \frac{1}{2}\Gamma\sigma\left(|u_1 + u_2|^2 + i|u_1 + iu_2|^2 - (1+i)|u_1|^2 - (1+i)|u_2|^2\right)$$

  is known.

- So given illuminations $\{g_j\}_{j=1}^2$, we can reconstruct from the measured internal data the new data:

$$E_j = \Gamma\sigma u_j u_1^*,$$

  for $j = 1, 2$. In particular, we know

$$\frac{u_2}{u_1} = \frac{E_2}{E_1}.$$

Reconstruction of (Γ, σ, η) (cont.)

Theorem

Let $\{g_j\}_{j=1}^2$ be a set of incident source functions, and suppose that the measured data $\{E_j\}_{j=1}^2$ satisfy the following two conditions:

1. $E_1(\mathbf{x}) \geq \alpha_0 > 0$ *for some* α_0, *a.e.* $\mathbf{x} \in \Omega$.
2. *The vector field*

$$\beta(\mathbf{x}) := \nabla \frac{E_2}{E_1}$$

is $W^{1,\infty}$ *field such that* $|\beta| \geq \beta_0 > 0$ *for some* β_0, *a.e.* $\mathbf{x} \in \Omega$.

Then Γ, η, *and* σ *are uniquely determined from the data* $\{E_j\}_{j=1}^2$.

Reconstruction of (Γ, σ, η) (cont.)

Proof.

Following Bal and Ren [3] (2011), we multiply the equation for u_1 by u_2 and multiply the equation for u_2 by u_1, and subtract to get

$$u_1 \Delta u_2 - u_2 \Delta u_1 = 0\,.$$

We can then rewrite this into

$$\nabla \cdot u_1^2 \nabla \frac{u_2}{u_1} = \nabla \cdot u_1^2 \nabla \frac{E_2}{E_1} = \nabla \cdot (u_1^2 \beta) = 0\,.$$

This is a transport equation for u_1. Together with the boundary condition $u_1 = g_1$ on $\partial\Omega$ and assumption 2, classical results show that there exists a unique weak solution u_1. Hence we can reconstruct

$$k^2(1+\eta) + ik\sigma = -\frac{\Delta u_1}{u_1}\,.$$

This gives us η and σ. The last step is to reconstruct Γ as $\Gamma = \frac{H_1}{\sigma |u_1|^2}$. □

Table of Contents

Higher-order linearization

- Recall the original system:

$$\begin{array}{rcll} \Delta u_\varepsilon + k^2(1+\eta)u_\varepsilon + ik\sigma u_\varepsilon & = & -k^2\gamma u_\varepsilon^* v_\varepsilon, & \text{in } \Omega \\ \Delta v_\varepsilon + (2k)^2(1+\eta)v_\varepsilon + i2k\sigma v_\varepsilon & = & -(2k)^2\gamma u_\varepsilon^2, & \text{in } \Omega \\ u_\varepsilon = \varepsilon g, & & v_\varepsilon = \varepsilon h, & \text{on } \partial\Omega\,. \end{array}$$

- Taking the 2nd derivative of the forward model with respect to ε at $\varepsilon = 0$ yields

$$\begin{array}{rcll} \Delta u^{(2)} + k^2(1+\eta)u^{(2)} + ik\sigma u^{(2)} & = & -2k^2\gamma u^{(1)*} v^{(1)}, & \text{in } \Omega \\ \Delta v^{(2)} + (2k)^2(1+\eta)v^{(2)} + i2k\sigma v^{(2)} & = & -2(2k)^2\gamma (u^{(1)})^2, & \text{in } \Omega \\ u^{(2)} = 0, & & v^{(2)} = 0, & \text{on } \partial\Omega\,. \end{array}$$

  Note that $u^{(1)}$ and $v^{(1)}$ are known.
- Meanwhile, the 3rd-order derivative of the data is

$$H^{(3)} = 3\Gamma\sigma\left(u^{(1)*}u^{(2)} + u^{(1)}u^{(2)*} + v^{(1)*}v^{(2)} + v^{(1)}v^{(2)*}\right)\,.$$

- If not for the conjugated terms $u^{(2)*}$ and $v^{(2)*}$ in $H^{(3)}$, these 3 equations would constitute a system of linear PDEs for $u^{(2)}$, $v^{(2)}$, γ.

- **Trick**: Take the complex conjugate of the equations for $u^{(2)}$ and $v^{(2)}$ to get the following system of linear PDEs for $(u^{(2)}, v^{(2)})$, $(u^{(2)*}, v^{(2)*})$, and γ:

$$\begin{aligned}
(\Delta + q_1)u^{(2)} + 2k^2 u^{(1)*} v^{(1)} \gamma &= 0, && \text{in } \Omega \\
(\Delta + q_1^*)u^{(2)*} + 2k^2 u^{(1)} v^{(1)*} \gamma &= 0, && \text{in } \Omega \\
(\Delta + q_2)v^{(2)} + 2(2k)^2 (u^{(1)})^2 \gamma &= 0, && \text{in } \Omega \\
(\Delta + q_2^*)v^{(2)*} + 2(2k)^2 (u^{(1)*})^2 \gamma &= 0, && \text{in } \Omega \\
u^{(1)*} u^{(2)} + u^{(1)} u^{(2)*} + v^{(1)*} v^{(2)} + v^{(1)} v^{(2)*} &= \frac{H^{(3)}}{3\Gamma\sigma}, && \text{in } \Omega \\
(u^{(2)}, u^{(2)*}, v^{(2)}, v^{(2)*}) &= (0,0,0,0), && \text{on } \partial\Omega\,,
\end{aligned}$$

  where we have used the notation

$$q_1 := k^2(1+\eta) + ik\sigma, \qquad q_2 := (2k)^2(1+\eta) + i2k\sigma\,.$$

- The rest of the proof consists of analyzing the uniqueness of the solution to this linear system. The analysis is based on the uniqueness theory for redundant elliptic systems reviewed in Bal [2] (2013).

Reconstruction of γ (cont.)

To guarantee uniqueness, we found it convenient to make the following assumptions:

1. In addition to the internal data, we also have access to the Dirichlet-to-Neumann map
$$\Pi_\gamma : (g, h) \mapsto \left(\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}, \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} \right).$$
2. The domain Ω is sufficiently small.

We believe that these assumptions can be removed without breaking the uniqueness and stability results.

Reconstruction of γ (cont.)

Theorem

When the domain Ω is sufficiently small, there exist illumination directions g and h in (2) such that

$$(H^{(3)}, J_u^{(2)}, J_v^{(2)}) = (\widetilde{H}^{(3)}, \widetilde{J}_u^{(2)}, \widetilde{J}_v^{(2)}) \implies \gamma = \widetilde{\gamma},$$

and $\forall p > 1$,

$$\|\gamma - \widetilde{\gamma}\|_{L^p(\Omega)} \leq C\Big(\Big\|H^{(3)} - \widetilde{H}^{(3)}\Big\|_{W^{2,p}(\Omega)} + \Big\|J_u^{(2)} - \widetilde{J}_u^{(2)}\Big\|_{W^{1-1/p,p}(\partial\Omega)} + \Big\|J_v^{(2)} - \widetilde{J}_v^{(2)}\Big\|_{W^{1-1/p,p}(\partial\Omega)} \Big),$$

for some constant $C = C(\Omega, \Gamma, \eta, \sigma) > 0$.

Table of Contents

Numerical experiments

- We perform numerical reconstructions with a standard least-squares optimization approach in which we find the coefficients $\eta, \sigma, \gamma, \Gamma$ by minimizing the functional

$$\Psi(\eta,\sigma,\gamma) := \frac{1}{2}\sum_{j=2}^{N} \left\| \frac{|u_j|^2 + |v_j|^2}{|u_1|^2 + |v_1|^2} - \frac{H_j}{H_1} \right\|_{L^2(\Omega)}^2 + \beta_1 \|\nabla \eta\|_{L^1(\Omega)} + \beta_2 \|\nabla \sigma\|_{L^1(\Omega)} + \beta_2 \|\nabla \gamma\|_{L^2(\Omega)},$$

  where we assumed that we collected data from N different boundary conditions. The regularization parameters $\beta_1, \beta_2, \beta_3$ are selected by a trial and error approach.
- The gradient of the objective function is computed using the standard adjoint state method.

- Reconstruction of γ

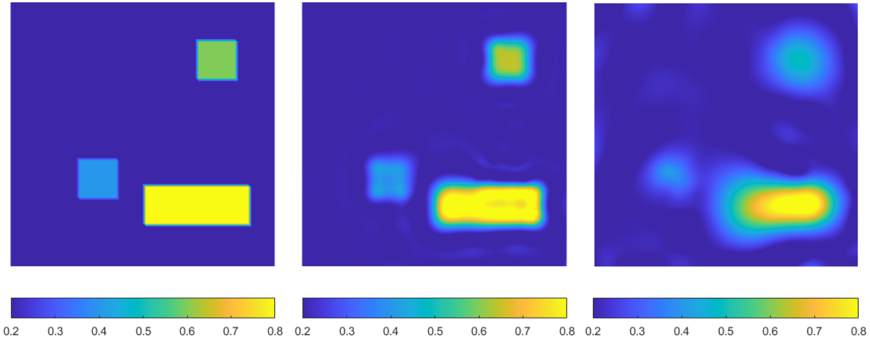

Figure 1: True γ (left), γ reconstructed from noise-free data (middle), and γ reconstructed from data containing 1% random noise (right).

- Simultaneous reconstruction of $\eta, \sigma, \gamma, \Gamma$

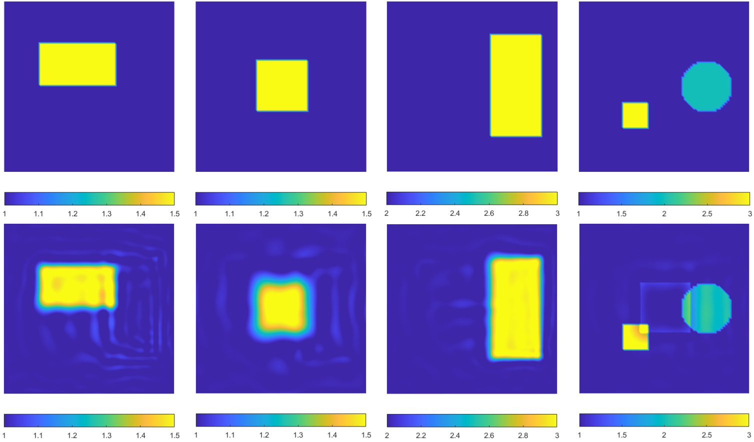

Figure 2: True (top) and reconstructed (bottom) $(\eta, \sigma, \gamma, \Gamma)$. From left to right are η, σ, γ, and Γ.

Future directions

- Well-posedness of the forward model for large boundary data?
- We have assumed Dirichlet boundary condition for the generated second harmonic wave v in the forward model. This should certainly be replaced with homogeneous Robin-type boundary conditions that are more physical.
- Try to remove the two assumptions in the reconstruction theorem for the second-order susceptibility parameter γ.
- The linearization method requires access to a one-parameter family of datasets generated from ε-dependent boundary source. This is a large amount of data. It would be of great interest to see if our uniqueness and stability results can be reproduced for a finite number of measurements.

References

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[3] G. Bal and K. Ren, *Multi-source quantitative PAT in diffusive regime*, Inverse Problems, 27 (2011). 075003.

[4] G. Bal, K. Ren, G. Uhlmann, and T. Zhou, *Quantitative thermo-acoustics and related problems*, Inverse Problems, 27 (2011). 055007.

[5] K. Ren and N. Soedjak, *Recovering coefficients in a system of semilinear Helmholtz equations from internal data*, arXiv:2307.01385, (2023).